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DOUBLY-PERIODIC PROBLEM OF THE THEORY OF ELASTICITY FOR AN ISOTROPIC MEDIUM WEAKENED BY CONGRUENT GROUPS OF ARBITRARY HOLES

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We consider the general doubly-periodic problem of the theory of elasticity for an isotropic medium when within the limits of the period-parallelogram we have a group of nonintersecting arbitrary holes. The problem reduces to a Fredholm integral equation of the second kind whose solvability will be proved. We consider also the problem of reduction for an anisotropic doubly-periodic lattice.

One of the doubly-periodic problems has been studied for the first time in [1]. Various classes of doubly-periodic problems in the extension and bending of lattices have been considered in [2]. Doubly-periodic problems when within the limits of the period-parallelogram there exists one hole of a general form are studied in [3, 4]. A lattice with groups of congruent circular holes has been studied in [5]. The solution of a series of doubly-periodic problems for physically nonlinear and also for anisotropic media have been given in [6−8]. The elasto-plastic problem for a regular isotropic lattice with circular holes has been studied in [9]. The general doubly-periodic problem for anisotropic media has been studied in [10], while the general formulation of the reduction problem for lattices is given in [11].

1. Let ω_1 and ω_2 $(\operatorname{Im}\omega_1 = 0,\ \operatorname{Im}(\omega_2/\omega_1) > 0)$ be the fundamental periods of the lattice. We will assume that within the limits of each period-parallelogram there exists a group of k nonintersecting holes of general form and that these groups are congruent with each other.

Let L_{00}^j $(j = 0, 1, \ldots, k-1)$ be the contour of the jth hole in the fundamental period parallelogram, $l_{00} = \bigcup L_{00}^j$, $L = \bigcup l_{mn}$. We denote by D the domain occupied by the lattice, the boundary of the domain is the totality of all L_{mn}^j, i.e. L. We assume that L_{00}^j $(j = 0, 1, \ldots, k-1)$ is a simple, smooth, closed contour. The finite continuum bounded by the curve L_{00}^j is denoted by D_{00}^j (Fig. 1).

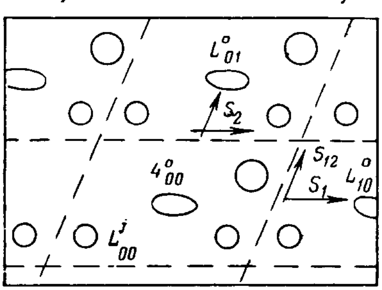

Fig. 1

By the first fundamental doubly-periodic problem for the described lattice we understand the boundary value problem regarding the determination of the stresses in D, when the same load acts on each of the congruent contours while within the limits of the period-parallelogram we have the mean stresses S_1, S_2 and S_{12} (Fig. 1).

The mean stresses on the areas perpendicular to the coordinate axes are expressed in terms of S_1, S_2 and S_{12} with the aid of the relations

$$\sigma_1 \sin\alpha = S_1 + 2S_{12}\cos\alpha + S_2\cos^2\alpha,\ \tau = S_{12} + S_2\cos\alpha,\ \sigma_2 = S_2\sin\alpha \quad (1.1)$$

The principal vector of the external loads on L_{00}^j ($j = 0, 1, \ldots, k-1$) is taken to be equal to zero. Under these conditions the distribution of the stresses in the lattice has a doubly-periodic character.

It is convenient to represent the Kolosov-Muskhelishvili functions $\varphi(z)$ and $\psi(z)$, which describe the states of stress and strain in the lattice, in such a way that they should reflect the doubly-periodic character of the problem. This can be suitably done by a modification of the known integral representations given in [12].

By assumption $\varphi(z)$ and $\bar{z}\varphi'(z) + \psi(z)$ must be quasi-periodic functions. Therefore we can write for $\varphi(z)$

$$\varphi(z) = \frac{1}{2\pi i}\int_{l_{00}} \omega(t)\,[\zeta(t-z) - \zeta(t)]\,dt + \sum_{j=0}^{k-1} b_j \zeta(z - z_j) + Az \quad (1.2)$$

Here $\zeta(z)$ is the Weierstrass zeta function, $\omega(t)$ is the desired density, $z_j \in D_{00}^j$ ($z_0 = 0$), the constants b_j are some functionals which will be given below. In (1.2) the integral term represents a quasi-periodic function. Integrals of this type have been studied in [13, 14]. A systematic investigation of the theory of integrals of the Cauchy type with automorphic kernels is contained in [15].

The structure of the function $\psi(z)$ is more complicated and for its construction we need, besides the integrals with kernels of the type of Weierstrass' zeta functions, some special integrals with regular kernels. We introduce the function [11]

$$\rho_1(z) = \sum_{mn}{}' \left\{\frac{\bar{P}}{(z-P)^2} - 2z\,\frac{\bar{P}}{P^3} - \frac{\bar{P}}{P^2}\right\}, \qquad P = m\omega_1 + n\omega_2 \quad (1.3)$$

This meromorphic function satisfies the following relations at the congruent points:

$$\begin{aligned} \rho_1(z+\omega_1) - \rho_1(z) &= \bar{\omega}_1\rho(z) + \gamma_1 \\ \rho_1(z+\omega_2) - \rho_1(z) &= \bar{\omega}_2\rho(z) + \gamma_2 \end{aligned} \quad (1.4)$$

Here $\rho(z)$ is the Weierstrass elliptic function while γ_1 and γ_2 are known constants [11]. We write down the representation for the function $\psi(z)$ which satisfies the required conditions

$$\psi(z) = \frac{1}{2\pi i}\int_{l_{00}} (\overline{\omega(t)}\,dt + \omega(t)\,\overline{dt})\,[\zeta(t-z) - \zeta(t)] -$$

$$-\frac{1}{2\pi i}\int_{l_{00}} \omega(t)\,[\bar{t}\rho(t-z) - \rho_1(t-z)]\,dt + \sum_{j=0}^{k-1} b_j\,[\zeta(z-z_j) + \rho_1(z-z_j)] + Bz \quad (1.5)$$

The constants b_j, which occur in the representations (1.2) and (1.5) are determined in the same way as in [12], except for an unessential change in the formulas

$$b_j = \frac{1}{2\pi i}\int_{L_{00}^j} \{\omega(t)\,\overline{dt} - \overline{\omega(t)}\,dt\}, \qquad j = 0, 1, \ldots, k-1 \quad (1.6)$$

The representations (1.2), (1.5) guarantee the quasi-periodicity of the functions $\varphi(z)$ and $\bar{z}\varphi'(z) + \psi(z)$. Indeed, from (1.2), taking into account the quasi-periodicity of the Weierstrass zeta function, we obtain

$$\varphi(z+\omega_\nu)-\varphi(z)=A\omega_\nu+b\delta_\nu, \qquad \nu=1,2 \tag{1.7}$$

$$b=\frac{1}{2\pi i}\int_{l_\infty}\{\omega(t)\,\overline{dt}-\omega(t)\,dt-\overline{\omega(t)}\,dt\},\ \delta_\nu=2\zeta\left(\frac{\omega_\nu}{2}\right)$$

Then, by virtue of the periodicity of $\varphi'(z)$, we have

$$[\bar{z}\varphi'(z)+\psi(z)]\,\Big|_z^{z+\omega_\nu}=\bar{\omega}_\nu\varphi'(z)+\psi(z+\omega_\nu)-\psi(z) \tag{1.8}$$

Substituting into (1.8) the functions φ, φ' and ψ from (1.2) and (1.5) and taking into account relation (1.4), we find

$$[\bar{z}\varphi'(z)+\psi(z)]\,\Big|_z^{z+\omega_\nu}=A\bar{\omega}_\nu+B\omega_\nu-a\delta_\nu+b\gamma_\nu, \qquad \nu=1,2$$

$$a=\frac{1}{\pi i}\int_{l_\infty}\overline{\omega(t)}\,dt \tag{1.9}$$

Formulas (1.7) and (1.9) prove the stated assertion.

We consider now the static conditions from which it is necessary to determine the constants A and B which occur in (1.2) and (1.5). We introduce the function

$$g(z)=\varphi(z)+z\overline{\varphi'(z)}+\overline{\psi(z)} \tag{1.10}$$

The principal vector of the forces acting along the arbitrary arc AB within the limits of the period-parallelogram, is given by the relation [16]

$$X+iY=-ig(z)\,\Big|_A^B \tag{1.11}$$

We have, taking into account (1.1) and (1.10), the static conditions

$$g(z+\omega_2)-g(z)=i(S_1+S_{12}e^{i\alpha})\,|\omega_2| \tag{1.12}$$
$$g(z+\omega_1)-g(z)=-i(S_{12}+S_2e^{i\alpha})\,\omega_1$$

Computing the increment of the function $g(z)$ in the congruent points, we arrive, by virtue of (1.7) and (1.9) to the following equations relative to the constants A and B:

$$(A+\bar{A})\,\omega_1+\bar{B}\bar{\omega}_1+\delta_1 b+\bar{\gamma}_1\bar{b}-\bar{a}\,\bar{\delta}_1=-i\omega_1(S_{12}+S_2e^{i\alpha})$$
$$(A+\bar{A})\,\omega_2+\bar{B}\,\bar{\omega}_2+\delta_2 b+\bar{\gamma}_2\bar{b}-\bar{a}\,\bar{\delta}_2=i\,|\omega_2|\,(S_1+S_{12}e^{i\alpha}) \tag{1.13}$$
$$[\delta_1\omega_2-\delta_2\omega_1=2\pi i,\ \ \gamma_2\omega_1-\gamma_1\omega_2=\delta_1\bar{\omega}_2-\delta_2\bar{\omega}_1]$$

The relations (1.13) contain four real equations relative to three unknown constants $\operatorname{Re}A$, $\operatorname{Re}B$ and $\operatorname{Im}B$. Multiplying the first of Eqs. (1.13) by ω_2 and the second one by ω_1 and subtracting them one from the other, we obtain, by taking into account the relations [11] given between the square brackets in (1.13), the expression for the constant B

$$B=\frac{\delta_1-\gamma_1}{\omega_1}b-\frac{2\pi}{S}\operatorname{Re}b-\left(\frac{\pi}{S}-\frac{\delta_1}{\omega_1}\right)\operatorname{Re}a-\frac{1}{2\sin\alpha}(S_1+2S_{12}e^{+i\alpha}+S_2e^{+2i\alpha}) \tag{1.14}$$

Here $S=\omega_1\operatorname{Im}\omega_2$ is the area of the period-parallelogram. Similarly, multiplying the first of Eqs. (1.13) by $\bar{\omega}_2$ and the second one by $\bar{\omega}_1$ and subtracting them one from the other, we find

$$\operatorname{Re}A=-\frac{1}{\omega_1}\operatorname{Re}(b\delta_1)+\frac{\pi}{S}\operatorname{Re}b+\frac{\pi}{2S}\operatorname{Re}a+\frac{1}{4\sin\alpha}(S_1+2S_{12}\cos\alpha+S_2) \tag{1.15}$$

The compatibility condition of the equalities (1.13) is the relation

$$\operatorname{Im} a = 0 \tag{1.16}$$

By virtue of (1.9) the condition (1.16) takes the form

$$\operatorname{Re} \int_{l_{00}} \omega(t)\, \overline{dt} = 0 \tag{1.17}$$

Thus, under the condition (1.17), the representations (1.2) and (1.5), with the constants $\operatorname{Re} A$ and B defined by the formulas (1.4) and (1.15), ensure the presence in the lattice of the given mean stresses S_1, S_2 and S_{12} and, consequently, they guarantee the vanishing of the principal vector and the principal moment of all forces acting along the boundary of the period-parallelogram. Obviously, the quantity $\operatorname{Im} A$ remains arbitrary.

2. The boundary condition of the first fundamental problem has the form [16]

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = f(t) + C^{j}_{mn}, \qquad j = 0, 1, \ldots, k-1 \tag{2.1}$$

$$f(t) = i\int_{t_0}^{t} (X_n + iY_n)\, dS, \quad C^{j}_{mn} = C_{00}{}^{j} + C_0 m + D_0 n, \quad m, n = 0, \pm 1, \pm, \ldots$$

where X_n, Y_n are the components of the load given on l_{00}, $t \in l_{00}$. Following [12], we define the constant C^{j}_{00} by the formula

$$C_{00}{}^{j} = -\int_{L_{00}{}^{j}} \omega(t)\, dS \tag{2.2}$$

Passing to the limiting values in (1.2), (1.5) by making use of the Sokhotski-Plemelj formulas and substituting them into the boundary condition (2.1), we obtain after some transformations a Fredholm integral equation of the second kind relative to the density $\omega(t)$

$$\omega(t_0) + \frac{1}{2\pi i}\int_{l_{00}} \omega(t)\, d\left\{\ln \frac{\sigma(t-t_0)\overline{\sigma(t)}}{\overline{\sigma(t-t_0)}\,\sigma(t)}\right\} + \frac{1}{2\pi i}\int_{l_{00}} \overline{\omega(t)}\, d\{\overline{\zeta_1(t-t_0)} - \tag{2.3}$$

$$-(t-t_0)\overline{\zeta(t-t_0)}\} + M\{\omega(t), t_0\} = F^*(t_0)$$

$$M\{\omega(t), t_0\} = \frac{1}{2\pi i}\int_{l_{00}} \overline{\omega(t)}\,\overline{\zeta(t)}\, dt + \sum_{j=0}^{k-1} b_j\,[2\operatorname{Re}\zeta(t_0 - z_j) +$$

$$+ \overline{\rho_1(t_0 - z_j)} - t_0\overline{\rho(t_0 - z_j)}] + t_0\left[\frac{2\pi}{S}\operatorname{Re} b - \frac{2}{\omega_1}\operatorname{Re}(b\delta_1) + \frac{\pi}{S}\operatorname{Re} a\right\} +$$

$$+ \bar{t}_0\left[\frac{\delta_1 - \bar{\gamma}_1}{\omega_1}\bar{b} - \frac{2\pi}{S}\operatorname{Re} b - \left(\frac{\pi}{S} - \frac{\bar{\delta}_1}{\omega_1}\right)\bar{a}\right] - C_{00}{}^{j}$$

$$F^*(t_0) = f(t_0) - \frac{t_0}{2\sin\alpha}(S_1 + 2S_{12}\cos\alpha + S_2) + \frac{\bar{t}_0}{2\sin\alpha}(S_1 + 2S_{12}e^{+i\alpha} + S_2 e^{+2i\alpha})$$

where $\sigma(z)$ is the Weierstrass sigma function and $\zeta_1(z)$ is determined by the relations

$$\zeta_1'(z) = -\rho_1(z), \qquad \zeta_1(0) = 0 \tag{2.4}$$

Thus, the problem reduces to the solving of Eq. (2.3) under the additional condition (1.16). We can eliminate the additional condition by adding the expression $\pi i t_0 \operatorname{Im}\bar{a}/S$ to the left-hand side of Eq. (2.3). Then every solution of the obtained modified equation will be also the solution of Eq. (2.3), satisfying (1.16), provided that the principal moment of the forces given on l_{00} is equal to zero. Indeed, replacing in (2.3) the term $M\{\omega(t), t_0\}$ by the expression $M\{\omega(t), t_0\} + \pi i t_0 \operatorname{Im}\bar{a}/S$, multiplying the equation thus obtained by dt_0 and integrating it along the contour l_{00}, we obtain, interchanging if necessary, the order of integration,

$$2i\ \mathrm{Im}\left\{\frac{1}{2\pi i}\int_{l_{00}} \overline{dt}_0 \int_{l_{00}} \omega(t)\,\zeta(t-t_0)\,dt - \frac{1}{2}\int_{l_{00}} \overline{\omega(t)}\,dt + \sum_{j=0}^{k-1} b_j \int_{l_{00}} \zeta(t-z_j)\,\overline{dt}\right\} + \\ + \left(2i\,\mathrm{Re}\,A - \frac{\pi}{S}\,\mathrm{Im}\,\overline{a}\right)\int_{l_{00}} (y\,dx - x\,dy) = \int_{l_{00}} f(t)\,\overline{dt} \qquad (2.5)$$

Since

$$\int_{l_{00}} (y\,dx - x\,dy) \neq 0$$

and since all the expressions in (2.5), except the term containing $\mathrm{Im}\,\bar{a}$, are pure imaginary quantities, we arrive at the required result.

3. Let us prove that the integral equation

$$\omega(t_0) + \frac{1}{2\pi i}\int_{l_{00}} \omega(t)\,d\left\{\ln\frac{\sigma(t-t_0)\,\overline{\sigma(t)}}{\overline{\sigma(t-t_0)}\,\sigma(t)}\right\} + \frac{1}{2\pi i}\int_{l_{00}} \overline{\omega(t)}\,d\,\{\overline{\zeta_1(t-t_0)} - \\ - (t-t_0)\,\overline{\zeta(t-t_0)}\} + M\{\omega(t),\,t_0\} + \frac{\pi i t_0}{S}\,\mathrm{Im}\,\bar{a} = F^*(t_0) \qquad (3.1)$$

is always solvable. To this end, we consider the homogeneous integral equation corresponding to Eq. (3.1) for $F^*(t_0) = 0$. Obviously, for $F^*(t) = 0$ it is necessary and sufficient that we have the conditions

$$S_1 = S_{12} = S_2 = 0, \qquad f(t) = 0$$

Thus, the homogeneous integral equation corresponds to the first fundamental problem of the theory of elasticity in the case of a zero external load.

We denote the solution of the homogeneous integral equation by $\omega_0(t)$. All functionals and functions which correspond to this solution will be assigned a zero as an upper or a lower index.

We have, according to (1.2) and (1.5)

$$\varphi_0(z) = \frac{1}{2\pi i}\int_{l_{00}} \omega_0(t)\,[\zeta(t-z) - \zeta(t)]\,dt + \sum_{j=0}^{k-1} b_j^0\zeta(z-z_j) + A_0 z$$

$$\psi_0(z) = \frac{1}{2\pi i}\int_{l_{00}} (\overline{\omega_0(t)}\,dt + \omega_0(t)\,\overline{dt})\,[\zeta(t-z) - \zeta(t)] - \qquad (3.2)$$

$$-\frac{1}{2\pi i}\int_{l_{00}} \omega_0(t)\,[\bar{t}\rho(t-z) - \rho_1(t-z)]\,dt + \sum_{j=0}^{k-1} b_j^0\,[\zeta(z-z_j) + \rho_1(z-z_j)] + B_0 z$$

The boundary condition for the functions $\varphi_0(z)$ and $\psi_0(z)$ acquires, according to (2.1), the form

$$\varphi_0(t) + t\overline{\varphi_0'(t)} + \overline{\psi_0(t)} = B_0^j \qquad (3.3)$$

$$B_0^j = -\int_{L_{00}^j} \omega_0(t)\,ds, \qquad j = 0, 1, \ldots, k-1$$

Consequently, $\varphi_0(z)$ and $\psi_0(z)$ describing the first fundamental problem for zero external forces, can be represented in the form

$$\varphi_0(z) = i\varepsilon z + c, \qquad \psi_0(z) = -\bar{d} \qquad (3.4)$$

In addition

$$B_0^j = c - d, \qquad j = 0, 1, \ldots, k-1 \qquad (3.5)$$

Comparing the corresponding increments of the functions from (3.2) and (3.4) at congruent points, we arrive at the equalities

$$A_0 = i\varepsilon,\ B_0 = 0,\ a_0 = \frac{1}{\pi i}\int_{l_{00}} \overline{\omega_0(t)}\,dt = 0,\ b_0 = \frac{1}{2\pi i}\int_{l_{00}} \{\omega_0\,\overline{dt} - \omega_0\,dt - \overline{\omega}_0\,dt\} = 0 \quad (3.6)$$

By virtue of (3.6), (3.4) and (3.2) we can write

$$\frac{1}{2\pi i}\int_{l_{00}} \omega_0(t)\,[\zeta(t-z) - \zeta(t)]\,dt + \sum_{j=0}^{k-1} b_j^0 \zeta(z - z_j) - c = 0, \qquad z \in D$$

$$\frac{1}{2\pi i}\int_{l_{00}} (\overline{\omega_0(t)}\,dt + \omega_0(t)\,\overline{dt})\,[\zeta(t-z) - \zeta(t)] - \quad (3.7)$$

$$-\frac{1}{2\pi i}\int_{l_{00}} \omega_0(t)\,[\bar{t}\rho(t-z) - \rho_1(t-z)]\,dt + \sum_{j=0}^{k-1} b_j^0\,[\zeta(z-z_j) + \rho_1(z-z_j)] + \bar{d} = 0$$

Integrating by parts in the second of the relations (3.7) and taking into account Cauchy's formula for each quasi-periodic function $F(z)$ with given cyclic weights α_1 and α_2 [14]

$$F(z) = \frac{1}{2\pi i}\int_{l_{00}} F(t)\,[\zeta(t-z) - \zeta(t)]\,dt - \frac{z}{2\pi i}(\alpha_1\delta_2 - \alpha_2\delta_1) \quad (3.8)$$

we can represent relations (3.7) in the form

$$\frac{1}{i}\,\Phi(z) = \frac{1}{2\pi i}\int_{l_{00}} \left[\omega_0(t) + \sum_{j=0}^{k-1} b_j^0 \zeta(t-z_j) - c\right][\zeta(t-z) - \zeta(t)]\,dt = $$

$$= \begin{cases} 0, & z \in D \\ \frac{1}{i}\,\varphi_*(z), & z \in D_{00}^{\,j} \end{cases} \quad (3.9)$$

$$\frac{1}{i}\,\Psi(z) = \frac{1}{2\pi i}\int_{l_{00}} \left[\overline{\omega_0(t)} - \bar{t}\omega_0'(t) + \sum_{j=0}^{k-1} b_j^0 \zeta(t-z_j) + e\right][\zeta(t-z) - \zeta(t)]\,dt +$$

$$+ Q(z) = \begin{cases} 0, & z \in D \\ \frac{1}{i}\,\psi_*(z), & z \in D_{00}^{\,j}, \quad j = 0, 1, \ldots, k-1 \end{cases} \quad (3.10)$$

Here

$$Q(z) = \frac{1}{2\pi i}\int_{l_{00}} \omega_0(t)\,\rho_1(t-z)\,dt + \sum_{j=0}^{k-1} b_j^0 \rho_1(z-z_j) \quad (3.11)$$

$$e = -\frac{1}{2\pi i}\int_{l_{00}} \bar{t}\omega_0(t)\,\rho(t)\,dt + \bar{d}$$

Evaluating the difference of the limiting values of the integral in (3.9), we find

$$i\Phi^-(t) = i\varphi_*(t) = \omega_0(t) + \sum_{j=0}^{k-1} b_j^0 \zeta(t-z_j) - c \quad (3.12)$$

It follows from here that the expression in the right-hand side of (3.12) is the boundary value of the functions $i\varphi_*(z)$, regular in the domains $D_{00}^{\,j}$ $(j = 0,1,\ldots, k-1)$.

If we substitute the expression for $\omega_0(t)$ from (3.12) into the formula for $Q(z)$ from (3.11), we obtain $Q(z) \equiv 0$. In this case we have

$$i\Psi^-(t) = i\psi_*(t) = \overline{\omega_0(t)} - \bar{t}\omega_0'(t) + \sum_{j=0}^{k-1} b_j^0 \zeta(t - z_j) + e \qquad (3.13)$$

i.e. the right-hand side of the formula (3.13) is the boundary value of the functions $i\psi_*(z)$, regular in D_{00}^j $(j = 0,1,\ldots, k-1)$. Eliminating, as usual, $\omega_0(t)$ from (3.12) and (3.13), we arrive at a functional relation on l_{00}

$$\overline{\varphi_*(t)} + \bar{t}\varphi_*'(t) + \psi_*(t) = i\sum_{j=0}^{k-1} b_j^0 [\overline{\zeta(t-z_j)} - \zeta(t-z_j) + \bar{t}\rho(t-z_j)] - i(\bar{c}+e) \quad (3.14)$$

We multiply (3.14) by dt and we integrate along each of the contours L_{00}^m, $(m = 0,1,\ldots$ $k-1)$. Taking into account that $\psi_*(t)$ is the boundary value of the function $\psi_*(z)$ regular in D_{00}^m, we obtain

$$\int_{L_{00}^m} \{\overline{\varphi_*(t)}\,dt - \varphi_*(t)\,\overline{dt}\} = -2\pi b_m^0 + i\sum_{j=0}^{k-1} b_j^0 \int_{L_{00}^m} [\overline{\zeta(t-z_j)}\,dt + \zeta(t-z_j)\,\overline{dt}]$$

From here we obtain the equalities

$$b_m = 0, \qquad m = 0,1,\ldots, k-1 \qquad (3.15)$$

Condition (3.14) acquires the form

$$\overline{\varphi_*(t)} + \bar{t}\varphi_*'(t) + \psi_*(t) = -i(e+\bar{c}) \qquad (3.16)$$

Consequently, the functions $\varphi_*(z)$ and $\psi_*(z)$ solve the first fundamental problem of the theory of elasticity for the domains D_{00}^j $(j = 0,1,\ldots, k-1)$ for a zero external load. We have

$$\varphi_*(z) = i\varepsilon_j z + c_j, \qquad \psi_*(z) = -\bar{d}_j, \qquad j = 0, 1, \ldots, k-1 \qquad (3.17)$$

The constants c_j, d_j and $e + c$ are connected by the relation $c_j - d_j - i(c + \bar{e}) = 0$. Expressing $\omega_0(t)$ from (3.12) and (3.17), we obtain

$$\omega_0(t) = t\varepsilon_j - c - ic_j \qquad (3.18)$$

Substituting $\omega_0(t)$ from (3.18) into the equality (3.15) and (1.6), we have

$$\varepsilon_j = 0, \qquad j = 0,1,\ldots, k-1 \qquad (3.19)$$

From the formulas (3.9), (3.10), (3.15), (3.18) and (3.19) we obtain the equalities

$$c_j = d_j = 0, \quad e + \bar{c} = 0, \quad j = 0,1,\ldots, k-1 \qquad (3.20)$$

Finally, using (3.5), (3.18), (3.20) and the fact that the constants B_0^j in (3.3) are independent of the index j, we obtain $c = d = 0$. Thus $\omega_0(t) = 0$ and the integral equation (3.1) has a unique solution.

4. We proceed now to the solving of the reduction problem for the generalized lattice under consideration. The meaning of this problem consists in the determination of the macroscopic elastic parameters of the lattice from the condition that the rigidity of the lattice under extension should coincide with the rigidity of some continuous anisotropic medium. According to the general formulation of the problems of this kind [11], it is necessary to identify the corresponding increments of the displacements in the congruent points of the lattice and in the continuous anisotropic medium.

Below we will consider only lattices which are symmetric with respect to the coordinate axes. The fundamental periods are taken in the form

$$\omega_2 = iH, \ \operatorname{Re}\omega_2 = 0 \ \ \operatorname{Im}\omega_1 = 0 \quad (\alpha = \pi/2) \qquad (4.1)$$

We introduce the notation

$$h(z) = 2G(u + iv) = \varkappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} \tag{4.2}$$

where u and v are the components of the displacement vector in the lattice and G is the shear modulus of the material of the lattice.

On the basis of formulas (1.7) and (1.8) we find the increments $h(z)$ in the congruent points

$$\Omega_1 = (\varkappa + 1)(A\omega_1 + b\delta_1) + (i\tau - \sigma_2)\omega_1 \tag{4.3}$$
$$\Omega_2 = (\varkappa + 1)(A\omega_2 + b\delta_2) - i(\sigma_1 \sin\alpha - \tau e^{-i\alpha} - i\sigma_2\cos\alpha)\,|\omega_2|$$

Equating the increments (4.3) with the corresponding increments of the function $h(z)$ in the continuous orthotropic medium for the same mean stresses σ_1, σ_2 and $\tau = 0$, we obtain a system of relations for the determination of the macroscopic elastic parameters E_1^*, E_2^*, μ_1^*, μ_2^*

$$(\varkappa + 1)(A\omega_1 + b\delta_1) - \sigma_2\omega_1 = \frac{E}{1+\mu}\,\frac{\sigma_1 - \mu_1^*\sigma_2}{E_1^*}\,\omega_1 \tag{4.4}$$
$$(\varkappa + 1)(A\omega_2 + b\delta_2) - i\sigma_1|\omega_2| = \frac{E}{1+\mu}\left(\frac{\sigma_1 - \mu_1^*\sigma_2}{E_1^*}\operatorname{Re}\omega_2 + i\,\frac{\sigma_2 - \mu_2^*\sigma_1}{E_2^*}\operatorname{Im}\omega_2\right)$$

According to (1.14) and (4.1), we can write

$$A\omega_1 + b\delta_1 = \left(\rho_1^* + \frac{\sigma_1 + \sigma_2}{4}\right)\omega_1, \qquad A\omega_2 + b\delta_2 = \left(\rho_2^* + \frac{\sigma_1 + \sigma_2}{4}\right)\omega_2$$
$$\rho_1^* = i\,\frac{\operatorname{Im}(b\delta_1)}{\omega_1} + \frac{\pi}{S}\operatorname{Re} b + \frac{\pi}{2S}\operatorname{Re} a$$
$$\rho_2^* = \frac{b\delta_2}{\omega_2} - \frac{\operatorname{Re}(b\delta_1)}{\omega_1} + \frac{\pi}{S}\operatorname{Re} b + \frac{\pi}{2S}\operatorname{Re} a$$

Introducing the notations

$$\rho_1^* = \rho_{11}\sigma_1 + \rho_{12}\sigma_2, \quad \rho_2^* = \rho_{21}\sigma_1 + \rho_{22}\sigma_2$$

and noting that the rigidity of the lattice does not depend on the mean stresses σ_1, σ_2, we obtain from (4.4)

$$\frac{E_1^*}{E} = (1 + 4\rho_{11})^{-1}, \qquad \frac{E_2^*}{E} = (1 + 4\rho_{22})^{-1}$$
$$\frac{E\mu_1^*}{E_1^*} = \mu - 4\rho_{12}, \qquad \frac{E\mu_2^*}{E_2^*} = \mu - 4\rho_{21}$$

The solvability condition of the problem of the reduction of a symmetric lattice to an equivalent medium is the equality $\rho_{12} = \rho_{21}$. In a similar way we can determine the macroscopic modulus of the second kind G^*.

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